Problem 6E,5
Suppose

$$X = \{0\} \bigcup \bigcup_{k=1}^{\infty} \{\frac{1}{k}\}$$

and $d(x, y) = |x - y|$.

- Show that (X, d) is a Banach space.
- Each set of the form $\{x\}$ is closed subset of $\mathbb{R}$ that has a nonempty interior as a subset of $\mathbb{R}$. Clearly X is a countably union of such sets. Explain why this does not violates the Baire's theorem.

Proof.

- This is easy to check.
- Note that each set of the form $\{x\}$ is closed subset of $\mathbb{R}$ and also closed subset of X. But if $x = \frac{1}{k}, k > 0$, as a subset of X, it has x as interior point.

□

Problem 6E,8
Suppose (X, d) is a complete metric space and $G_1, G_2, \ldots$ is a sequence of open dense subsets of X. Prove that $\bigcap_{k=1}^{\infty} G_k$ is dense subset of X.

Proof. Let U be a open subset of X and we need to show that $\bigcap_{k=1}^{\infty} G_k \bigcap U$ is nonempty. Since G_1 is open dense, we can find $\bar{B}(f_1, r_1) \subset G_1 \bigcap U, r_1 \in (0, 1)$. Now we can follow the proof of 6.76 (b) to find $f \in \bigcap_{k=1}^{\infty} G_k$, which shows that $\bigcap_{k=1}^{\infty} G_k \bigcap U$ is nonempty. □

Problem 6E,9
Prove that there dose not exists infinite-dimensional Banach space with a countable basis.

Proof. Otherwise, let B be a Banach space with countable basis $b_1, b_2, \ldots$. Set

$$B_n = \{b \in B | b \textit{ can be written as } \sum_{k=1}^{n} c_k b_k, |c_k| \leq n\}.$$

Then B_n is closed subset of B and $B = \bigcup_{n=1}^{\infty} B_n$. Then we know that there exists some n_0 such that $B(f, r) \subset B_{n_0}$ for some $f \in B, r > 0$. Thus $B(0, r)$ lies in some finite dimensional subspace and so is B, which is an obviously contradiction. □

Problem 6E,16
Suppose V is a Banach space with norm $\|.\|$ and $\phi : V \to F$ be a linear functional . Define another norm $\|.\|_\phi$ on V by

$$\|f\|_\phi = \|f\| + |\phi(f)|.$$

Prove that if V is a Banach space with norm $\|.\|_\phi$, then ϕ is continuous functional on V with the original norm.

Proof. Consider the map $I : (V, \|.\|_\phi) \to (V, \|.\|)$ by sending f to f. This is a one-one map between Banach space so by 6.83 this map has a bounded inverse. Then there exists some constant $C > 1$ such that for any $f \in V$,

$$\|f\|_\phi \leq C\|f\|.$$

This shows ϕ is continuous functional on V with the original norm. □

Problem 7A,5
Suppose (X, S, μ) is measure space and $1 < p < \infty, f \in L^p(X), h \in L^{p'}(X)$. Show tha equality holds in Holder inequality if and only if there exist nonnegative numbers a, b, not both 0, such that for almost every x,

$$a|f(x)|^p = b|h(x)|^{p'}.$$

Proof. The "if" part is obviouly. If the equality holds, we only need to consider the special case $\|f\|_p = \|h\|_{p'} = 1$.Note that the equality case for Young's inequality in 7.8 is $a^p = b^{p'}$. From the proof of 7.9, we know that for almost every x,

$$|f(x)|^p = |h(x)|^{p'}.$$

The general case follows similarly. □

Problem 7A,7
Suppose (X, S, μ) is measure space and $f, h : X \to F$ are measurable function. Prove that if for positive $p, q, r, \frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, then

$$\|fh\|_r \leq \|f\|_p \|h\|_q$$

Proof. Note that $\frac{r}{p} + \frac{r}{q} = 1$ thus $\frac{p}{r}, \frac{q}{r}$ is conjugate. Then apply the Holder inequality to function f^r, h^r gives the result. □

Problem 7A,11
Show that $\bigcap_{p>1} l^p \neq l^1$.

Proof. Note that $l^1 \subset \bigcap_{p>1} l^p$. But conversely, look at $a = (\frac{1}{k})_{k=1}^{\infty}$. Then $a \in \bigcap_{p>1} l^p - l^1$. □

Problem 7A,17
Suppose μ is a measure ,$1 < p \leq \infty$ and $f \in L^p$. Prove that for every $\epsilon > 0$, there exists simple function g such that $\|f - g\|_p < \epsilon$.

Proof. Assume first $1 \leq p < \infty$. First we consider positive f. By 2.89, we can find a sequence of simple function $f_n \leq f$ and f_n converges to f pointwisely. Thus by dominate convergence theorem,

$$\lim_{k\to\infty} \int |f_n - f|^p d\mu = \int \lim_{k\to\infty} |f_n - f|^p d\mu = 0.$$

Thus we can find the required function. For general f, we can work separately with its positive part and negative part.
For the case of $p = \infty$, we can use 2.89 directly. □